Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 9, 10

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Textbook Sections: 7.2, 7.3

Todays Topic: Trig Integrals (in Lecture 10, trig substitution...)

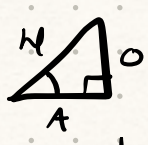
This week we'll dig into two new techniques for doing integrals:

1. Trig substitutions
2. Partial Fractions

In the first two lectures, we'll focus on (1), and we'll start today by getting into the swing of using lots of trig identities. Consider this lecture a "warm-up" for wednesday.

First recall the following extremely useful facts:

1. SOH-CAH-TOA



2. $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, $\cot(x) = \frac{1}{\tan x}$
3. $\cos^2 x + \sin^2 x = 1$
4. $$\begin{cases} \cos(x+y) = \cos x \cos y - \sin x \sin y \\ \sin(x+y) = \cos x \sin y + \cos y \sin x \end{cases}$$

Useful Trick: Do derive these pesky angle sum formulas, we can use the following clever idea:

Recall that $e^{ix} = \cos x + i \sin x$ for any $x \in \mathbb{R}$, where $i = \sqrt{-1}$.

Then

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

$$\| $$

$$e^{ix}e^{iy} = (\cos x + i\sin x)\cdot(\cos y + i \sin y) = \begin{matrix} \cos x \cos y - \sin x \sin y \\ + i(\cos x \sin y + \cos y \sin x) \end{matrix}$$

Keeping these identities in mind, lets try computing some integrals:

Goal 1: Compute integrals of the form $\int \sin^m x \cdot \cos^n x \, dx$

Goal 2: Compute integrals of the form $\int \tan^m x \cdot \sec^n x \, dx$

Exercise: After this lecture, try this out for $\int \cot^m x \csc^n x \, dx$

Towards Goal 1: It ends up being easier to handle this in cases.

- **Case 1:** m is odd, $m = 2k+1$
- **Case 2:** n is odd, $n = 2k+1$
- **Case 3:** m and n are even.

} note
- $m=0$, $n=1$
- $m=1$, $n=0$
- $m=n=1$

are easy!

Lets go case by case, using the identities above.

Case 1: $m = 2k+1$ is odd.

(i) Pull off a factor of sine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int \sin^{2k} x \cos^n x \sin x \, dx$$

(ii) use the identity (3):

$$\int \sin^{2k} x \cos^n x \sin x \, dx = \int (1-\cos^2 x)^k \cos^n x \sin x \, dx$$

(iii) use u-sub: $u = \cos x$ $\quad du = -\sin x \, dx$

$$\int (1-\cos^2 x)^k \cos^n x \sin x \, dx = -\int (1-u)^k u^n \, du$$

(iv) expand and compute ...

Case 2: $n = 2k+1$ is odd.

(i) Pull off a factor of cosine

(ii) use identity (3)

(iii) use u-sub $u = \sin x$ $\quad du = \cos x \, dx$

(iv) expand and compute

Case 3: m and n are even

(i) In this case, the above strategies based on u-substitutions fail, and we need another trick. This time, we leverage the following identities which follow from (3) & (4):

$$\begin{cases} \sin^2 x = \frac{1}{2}(1-\cos 2x) \\ \cos^2 x = \frac{1}{2}(1+\cos 2x) \\ \sin x \cos x = \frac{1}{2}\sin 2x \end{cases}$$

($\cos 2x = \cos^2 x - \sin^2 x$ apply (3))

(ii) Use these identities to reduce the problem to easier cases — don't be afraid to repeat these steps/cases multiple times!

A Helpful Note: using identies (3), we have the additional useful facts:

- $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

This allows us to integrate functions like

$\sin 4x \sin(2x)$, $\cos x \sin 5x$, $\cos(2x)\cos(3x)$, etc

Examples:

(1) $\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x = -\int (1-u^2)\,du = -u + \frac{1}{3}u^3 + C$ $(m=3, n=0)$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

(2) $\int_0^{\pi/2} \sin^5 x \cos^3 x\, dx = \int_0^{\pi/2} \sin^5 x (1-\sin^2 x)\cos x = \int_0^1 u^5(1-u^2)\,du = \cdots$ $u = \sin x$

(3) $\int \sin^2 x \cos^2 x\, dx = \int \sin^2 x (1-\sin^2 x) = \int \sin^2 x - (\sin^2 x)^2\, dx$

$= \int \frac{1}{2}(1-\cos 2x)\,dx - \int \frac{1}{4}(1-\cos 2x)^2\, dx$

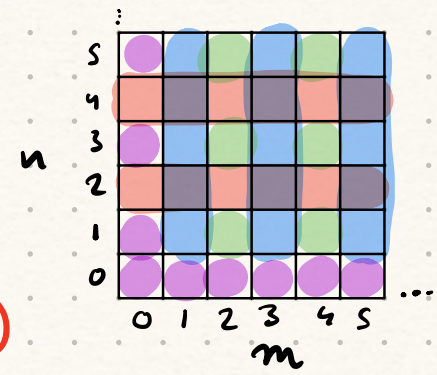
$= (\cdots) - \frac{1}{4}\int (1+\cos^2(2x) - 2\cos 2x)\,dx$

$= (\cdots) - \frac{1}{4}\int \cos^2(2x)\,dx = (\cdots) - \frac{1}{8}\int (1+\cos(4x))\,dx$

or $= \int (\sin x \cos x)^2 dx = \frac{1}{4}\int \sin^2(2x)\,dx = \frac{1}{4}\int (1-\cos(4x))^2\,dx = \cdots$

Towards Goal 2: $\int \tan^m x \sec^n x \, dx$

Again, it ends up being easier to handle this in cases:



Case 1: m is odd, $m = 2k+1$, $n \geq 1$

Case 2: $n \geq 2$ is even

Case 3: m is even, n is odd.

Case 4: The other edge cases (somewhat ad hoc)

Key Idea: use the following facts:

(5) $1 + \tan^2 x = \sec^2 x$ (divide (3) by $\cos^2 x$)

(6) $\frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \sec^2 x$

(7) $\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$

Case 1: m is odd, $m = 2k+1$

(i) Pull off a factor of $\tan(x)$ to pair with a factor of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int \tan^{2k}(x) \sec^{n-1}(x) \cdot (\tan x \sec x) \, dx$$

(ii) use the identity (5):

$$\int \tan^{2k}(x) \sec^{n-1}(x) \cdot (\tan x \sec x) \, dx = \int (\sec^2 x - 1)^k \sec^{n-1}(x) \cdot (\tan x \sec x) \, dx$$

(iii) use u-sub: $u = \sec x$ $du = (\tan x \sec x) \, dx$

$$\int (\sec^2 x - 1)^k \sec^{n-1}(x) \cdot (\tan x \sec x) \, dx = \int (u-1)^k u^{n-1} \, du$$

(iv) expand and compute ...

Case 2: $n \geq 2$ is even

(i) Pull off a factor of $\sec^2(x)$ (this is why we need $n \geq 2$!)

(ii) use (5) to convert the remaining $\sec^2 x$ factors to $(\tan^2(x) + 1)$'s

(iii) use the u-sub $u = \tan x$, $du = \sec^2 x \, dx$

(iv) Expand and compute.

Here are some of the important ad hoc cases not covered:

1) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \ln|\sec x| + C$ (u-sub)

2) $\int \sec x \, dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{1}{u} du = \ln|u| + C$
$= \ln|\sec x + \tan x| + C$

if we use $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x) dx$

Case 3: m even, n is odd.

(i) Use (5) to convert all $\tan^2 x$ factors to $(\sec^2 x - 1)$'s

(ii) Use other techniques to handle the integral — eg IBP

Ex: $\int \sec^3 x = \int \sec(x)(\sec^2 x \, dx) = (\sec x \tan x) - \int \sec x \tan^2 x \, dx$
$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$\rightarrow \int \sec^3 x \, dx = \frac{1}{2}\left[\sec x \tan x + \int \sec x \, dx\right]$
(from above).

Examples:

1) $\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x (\tan x \sec x) dx$
$= \int (u^2 - 1) u^2 \, du$

2) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

3) $\int \tan^8 x \sec^4 x \, dx = \int \tan^8 x \cdot (1 + \tan^2 x) \sec^2 x \, dx$
$= \int u^8 (1 + u^2) \, du = \cdots$

4) $\int \tan^3 x \, dx = \int (\sec^2 x - 1) \tan x = \frac{1}{2}\tan^2 x - \ln|\sec x| + C$